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Department of Mechanical and Aerospace Engineering

Data-Driven Control via Semidefinite Programming in Nonlinear Systems

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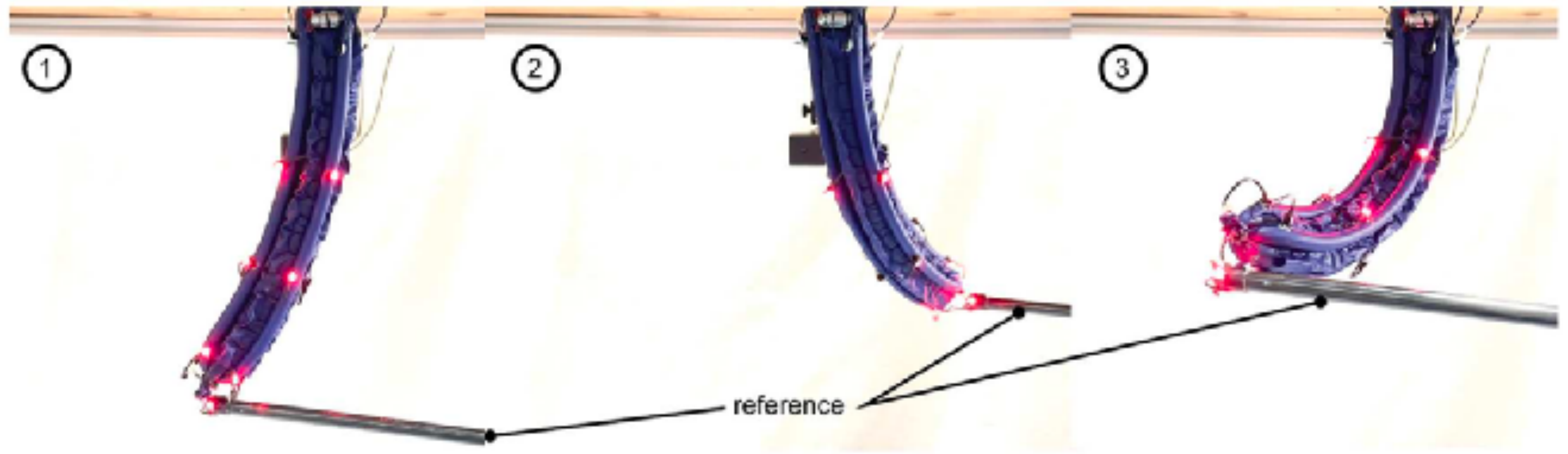
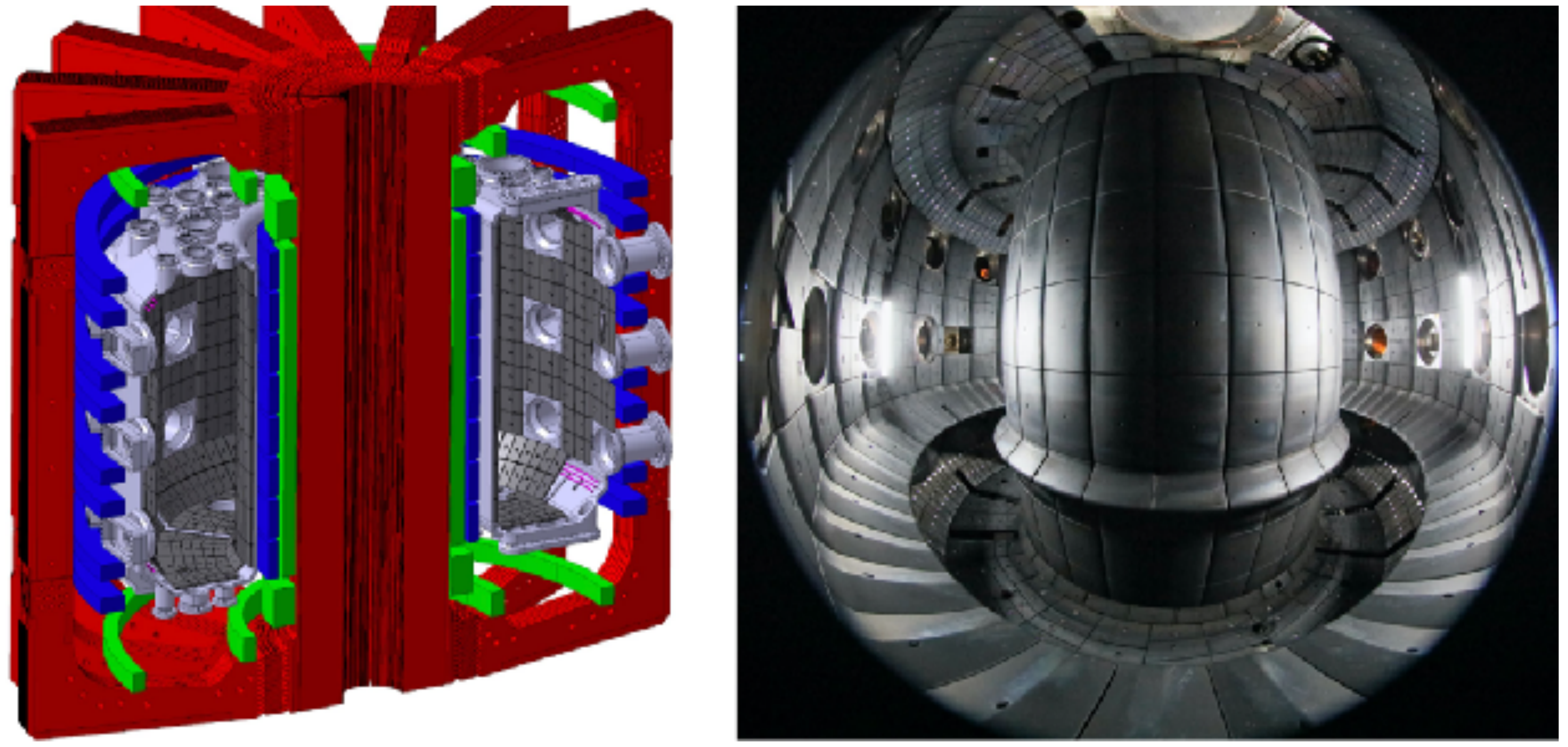
Data-Driven Frameworks for Control

- Data-driven control methods are useful for tackling complex engineering systems
    - Working with data vs. complex models
- Soft robotics, tokamaks for fusion energy, power grids have already seen success[1,2,3]
- Developing these methods rigorously is important for both intelligent and practical application
    - Semidefinite programming can leverage assumptions about the system for stability guarantees



Pneumatically actuated soft arm[1]



Tokamak à Configuration Variable[2]

[1] Haggerty, D. A., Banks, M. J., Kamenar, E., Cao, A. B., Curtis, P. C., Mezić, I., & Hawkes, E. W. (2023). Control of soft robots with inertial dynamics. *Science Robotics*, *8*(81)

[2] Degrave, J., Felici, F., Buchli, J., Neunert, M., Tracey, B., Carpanese, F., Ewalds, T., Hafner, R., Abdolmaleki, A., De Las Casas, D., Donner, C., Fritz, L., Galperti, C., Huber, A., Keeling, J., Tsimpoukelli, M., Kay, J., Merle, A., Moret, J.-M., … Riedmiller, M. (2022). Magnetic control of tokamak plasmas through deep reinforcement learning. *Nature*, *602*(7897), 414–419

[3] Yuan, Z., Zhao, C., & Cortés, J. (2024). Reinforcement learning for distributed transient frequency control with stability and safety guarantees. *Systems & Control Letters*, 185

Sum of Squares Polynomials

Definition (Sum of squares polynomials): A polynomial $p(x)$ of degree $2d$ ($d \in \mathbb{Z}_{>0}$) is sum of squares (SOS) if there exist polynomials $g_1(x), \ldots, g_k(x)$ s.t.

$$p(x) = \Sigma_{i=1}^{k} g_i(x)^2$$

$p(x)$ SOS $\implies p(x)$ nonnegative $\forall x \neq 0$

Theorem (Representation via PSD matrices) [4]: A polynomial $p(x)$ of degree $2d$ is SOS if and only if there exists a matrix $Q \succ 0$ of appropriate dimension such that with $m(x)$ vector consisting of all monomials in x of degree at most d,

$$p(x) = m(x)^\top Q m(x)$$

Ex: $$2x_1^2 + 2x_1x_2 + 2x_2^2 = [x_1 \quad x_2] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

[4] Nesterov, Y. (2000). Squared Functional Systems and Optimization Problems. In H. Frenk, K. Roos, T. Terlaky, & S. Zhang (Eds.), *High Performance Optimization* (Vol. 33, pp. 405–440)

Semidefinite Programs for Data-Driven Control of Linear Systems

Given discrete-time linear system we would like to stabilize to origin:

$$x(t+1) = Ax(t) + Bu(t)$$

Suppose A, B unknown, but snapshot/trajectory data can be generated:

$$\mathscr{U} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}$$

$$\mathscr{X}_- = \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}$$

$$\mathscr{X}_+ = \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}$$

Theorem [5]: The data $(\mathscr{U}, \mathscr{X}_-, \mathscr{X}_+)$ are informative for stabilization by state feedback if and only if there exists a matrix $\Theta \in \mathbb{R}^{T\times n}$ satisfying:

$$\mathscr{X}_-\Theta = (\mathscr{X}_-\Theta)^\top \text{ and } \begin{bmatrix} \mathscr{X}_-\Theta & \mathscr{X}_+\Theta \\ \Theta^\top \mathscr{X}_+^\top & \mathscr{X}_-\Theta \end{bmatrix} \succ 0$$

Moreover, $K = \mathscr{U}\Theta(\mathscr{X}_-\Theta)^{-1}$ will stabilize the system.

[5] van Waarde, H. J., Eising, J., Trentelman, H. L., & Camlibel, M. K. (2020). Data Informativity: A New Perspective on Data-Driven Analysis and Control. *IEEE Transactions on Automatic Control*, *65*(11), 4753–4768. IEEE Transactions on Automatic Control.

How Assumptions on Underlying Dynamics Grant Stability Guarantees

Model-based control design:

System matrices (A, B) → Choose K such that $(A + BK)$ Hurwitz/Schur →

$$P(A + BK) + (A + BK)^\top P \prec 0$$

or

$$(A + BK)^\top P(A + BK) - P \prec 0$$

Data-based control design:

Trajectory data $(\mathscr{U}, \mathscr{X}_-, \mathscr{X}_+)$

Parameterize $(A + BK)$ with data and solve requirements with data

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Data Efficiency of Direct Control Design

- The previous result shows how data can be used to directly design control, **bypassing model identification**
  - The method searches for a gain matrix K that stabilizes the collection of systems consistent with the data
  - **Requires less data** than identification
- Willems's fundamental lemma and persistency of excitation[6,7] give a precise answer for how the data should be collected

[6] Willems, J. C., Rapisarda, P., Markovsky, I., & De Moor, B. L. M. (2005). A note on persistency of excitation. *Systems & Control Letters*, *54*(4), 325–329.

[7] De Persis, C., & Tesi, P. (2020). Formulas for Data-Driven Control: Stabilization, Optimality, and Robustness. *IEEE Transactions on Automatic Control*, *65*(3), 909–924. IEEE Transactions on Automatic Control.

Problem Setup for Nonlinear Case

Given nonlinear system where dynamics unknown, but we can collect data:

$$\dot{x}(t) = f(x) + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \qquad (1)$$

$$\mathcal{U} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}$$

$$\mathcal{X}_- = \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}$$

$$\mathcal{X}_d = \begin{bmatrix} \dot{x}(0) & \dot{x}(1) & \cdots & \dot{x}(T-1) \end{bmatrix}$$

Assume we have a dictionary of functions $Z(x) : \mathbb{R}^n \to \mathbb{R}^S$ that can be used to represent $f(x)$:

$$\dot{x}(t) = AZ(x) + Bu$$

(where $A \in \mathbb{R}^{n \times S}$ unknown)

Can compute "lifted" data:

$$\mathcal{Z}_- = \begin{bmatrix} Z(x(0)) & Z(x(1)) & \cdots & Z(x(T-1)) \end{bmatrix}$$

Data is consistent with the equation:

$$\mathcal{X}_d = A\mathcal{Z}_- + B\mathcal{U}$$

[8] Guo, M., De Persis, C., & Tesi, P. (2020). Learning control for polynomial systems using sum of squares relaxations. *2020 59th IEEE Conference on Decision and Control (CDC)*, 2436–2441.

Design of Control and Lyapunov Function

Assume control has form $u = KZ(x)$. Closed loop system is:

$$\dot{x} = (A + BK)Z(x)$$

Consider candidate Lyapunov function:

$$V(x) = Z(x)^\top PZ(x) \text{ with } P \succ 0$$

$$\dot{V}(x) = \left(\frac{\partial Z(x)}{\partial x}(A + BK)Z(x)\right)^\top PZ(x) + Z(x)^\top P\left(\frac{\partial Z(x)}{\partial x}(A + BK)Z(x)\right)$$

$$= Z(x)^\top \left[\left(\frac{\partial Z(x)}{\partial x}(A + BK)\right)^\top P + P\frac{\partial Z(x)}{\partial x}(A + BK)\right] Z(x)$$

[8] Guo, M., De Persis, C., & Tesi, P. (2020). Learning control for polynomial systems using sum of squares relaxations. *2020 59th IEEE Conference on Decision and Control (CDC)*, 2436–2441.

Rewriting Expressions in Terms of Data

$$\dot{V}(x) = Z(x)^\top \left[\left(\frac{\partial Z(x)}{\partial x}(A+BK) \right)^\top P + P \frac{\partial Z(x)}{\partial x}(A+BK) \right] Z(x)$$

To write in terms of data, assume that the data $\mathscr{Z}\_$ is full row rank:

$$\exists G \in \mathbb{R}^{T \times S} \text{ s.t. } I_S = \mathscr{Z}\_G$$

Further, choose $K = \mathscr{U}G$

Then, the data can parameterize the closed loop dynamics of the system under this feedback:

$$A + BK = [B \quad A] \begin{bmatrix} K \\ I_n \end{bmatrix} = [B \quad A] \begin{bmatrix} \mathscr{U} \\ \mathscr{Z}\_ \end{bmatrix} G = \mathscr{X}_d G$$

[8] Guo, M., De Persis, C., & Tesi, P. (2020). Learning control for polynomial systems using sum of squares relaxations. *2020 59th IEEE Conference on Decision and Control (CDC)*, 2436–2441.

Tighted Conditions for Negative Definiteness

$$\dot{V}(x) = Z(x)^\top P \left[P^{-1} G^\top \mathcal{X}_d^\top \left(\frac{\partial Z(x)}{\partial x} \right)^\top + \frac{\partial Z(x)}{\partial x} \mathcal{X}_d G P^{-1} \right] P Z(x)$$

Then to write conditions for design of stabilizing control, we wish to impose negative definiteness of $\dot{V}(x)$ and search for valid matrices G, P satisfying all previous conditions

However, solving this computationally is not trivial due to the bilinearity in G, P

A common sufficient requirement is to impose negative definiteness of:

$$P^{-1} G^\top \mathcal{X}_d^\top \left(\frac{\partial Z(x)}{\partial x} \right)^\top + \frac{\partial Z(x)}{\partial x} \mathcal{X}_d G P^{-1}$$

[8] Guo, M., De Persis, C., & Tesi, P. (2020). Learning control for polynomial systems using sum of squares relaxations. *2020 59th IEEE Conference on Decision and Control (CDC)*, 2436–2441.

Sufficient Conditions for Nonlinear Stabilization

$$-P^{-1}G^\top \mathcal{X}_d^\top \left(\frac{\partial Z(x)}{\partial x}\right)^\top - \frac{\partial Z(x)}{\partial x}\mathcal{X}_d G P^{-1} \succ 0, \quad \forall x \in \mathbb{R}^n$$

Letting $Y = GP^{-1}$, we can collect the results in the following theorem:

Theorem [8]**:** For the nonlinear system (1), assuming $Z(x) = 0 \iff x = 0$ and full row rank data matrix $\mathcal{Z}\_$, if there exist matrices $Y \in \mathbb{R}^{T\times S}, P \in \mathbb{R}^{S\times S}, P \succ 0$ and SOS poly $\epsilon(x)$ such that:

1. $\mathcal{Z}\_Y = P^{-1}$ (because $\mathcal{Z}\_Y = \mathcal{Z}\_GP^{-1} = I_S P^{-1} = P^{-1}$)

2. $-Y^\top \mathcal{X}_d^\top \left(\frac{\partial Z(x)}{\partial x}\right)^\top - \frac{\partial Z(x)}{\partial x}\mathcal{X}_d Y - \epsilon(x) I_S$ is an SOS matrix polynomial

Then $u(x) = \mathcal{U}YP^{-1}Z(x) = \mathcal{U}GZ(x)$ stabilizes the polynomial system globally.

[8] Guo, M., De Persis, C., & Tesi, P. (2020). Learning control for polynomial systems using sum of squares relaxations. *2020 59th IEEE Conference on Decision and Control (CDC)*, 2436–2441.

Key Assumptions That Obstruct Necessity

- The previous result is a sufficient result relies on:
  - Assuming there will exist Lyapunov functions of form $Z(x)^\top PZ(x)$
  - Tightening negative definiteness of $Z(x)^\top P\left[H(x)\right]PZ(x)$ to $H(x) \prec 0 \quad \forall x$
- Essentially, **if the method fails, nothing is known about whether or not the system was stabilizable** via different approaches
- Necessary conditions in the form of feasibility guarantees are critical
  - Classes of nonlinear systems where any challenges resolved?

Systems that Permit Linear Embeddings

Consider nonlinear systems that permit linear embeddings, where $\dot{x} = f(x) = AZ(x)$ and $\exists \phi(x) : \mathbb{R}^n \to \mathbb{R}^W$ such that $\dot{\phi}(x) = \tilde{A}\phi(x)$ (where both $\phi(x), \tilde{A}$ unknown)

Assuming $\phi(x) = TZ(x), \quad T \in \mathbb{R}^{W \times S}$ (our dictionary contains the components of $\phi(x)$):

$$\dot{\phi}(x) = T\dot{Z}(x) = T\frac{\partial Z(x)}{\partial x}AZ(x)$$

$$= \tilde{A}\phi(x) = \tilde{A}TZ(x)$$

$$\implies Z(x) \in \mathcal{N}\left(\tilde{A}T - T\frac{\partial Z(x)}{\partial x}A\right)$$

$$\exists P \text{ s.t. } \dot{V}(x) = Z(x)^\top \left[A^\top \left(\frac{\partial Z(x)}{\partial x}\right)^\top P + P\frac{\partial Z(x)}{\partial x}A\right] Z(x) = Z(x)^\top \left[-T^\top T\right] Z(x)$$

However, issues persist…

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Exponentially Stable Polynomial Systems Offer Useful Converse Lyapunov Theorems

- Polynomial systems that are exponentially stable on compact sets have Lyapunov functions that are SOS polynomials and have bounded degree[9]
    - This addresses some aspects of the dictionary question and Lyapunov functions of the form $Z(x)^\top P Z(x)$
    - However, no guarantees given on whether or not $-\dot{V}(x)$ is SOS[10] and once again, $Z(x)^\top [H(x)] Z(x) > 0 \not\Rightarrow H(x) \succ 0$

[9] Peet, M. M., & Papachristodoulou, A. (2012). A Converse Sum of Squares Lyapunov Result With a Degree Bound. *IEEE Transactions on Automatic Control*, *57*(9), 2281–2293.

[10] Ahmadi, A. A., & Parrilo, P. A. (2011). Converse results on existence of sum of squares Lyapunov functions. *IEEE Conference on Decision and Control and European Control Conference*, 6516–6521.

Concluding Remarks

- Data-driven approaches for control have shown promise in designing control for systems that can be difficult to model
- However, developing these methods with the same rigor as in model-based control is critical
- While semidefinite programming has yielded insights into how data-driven control can be characterized for linear systems, **extensions to nonlinear systems remain very challenging**

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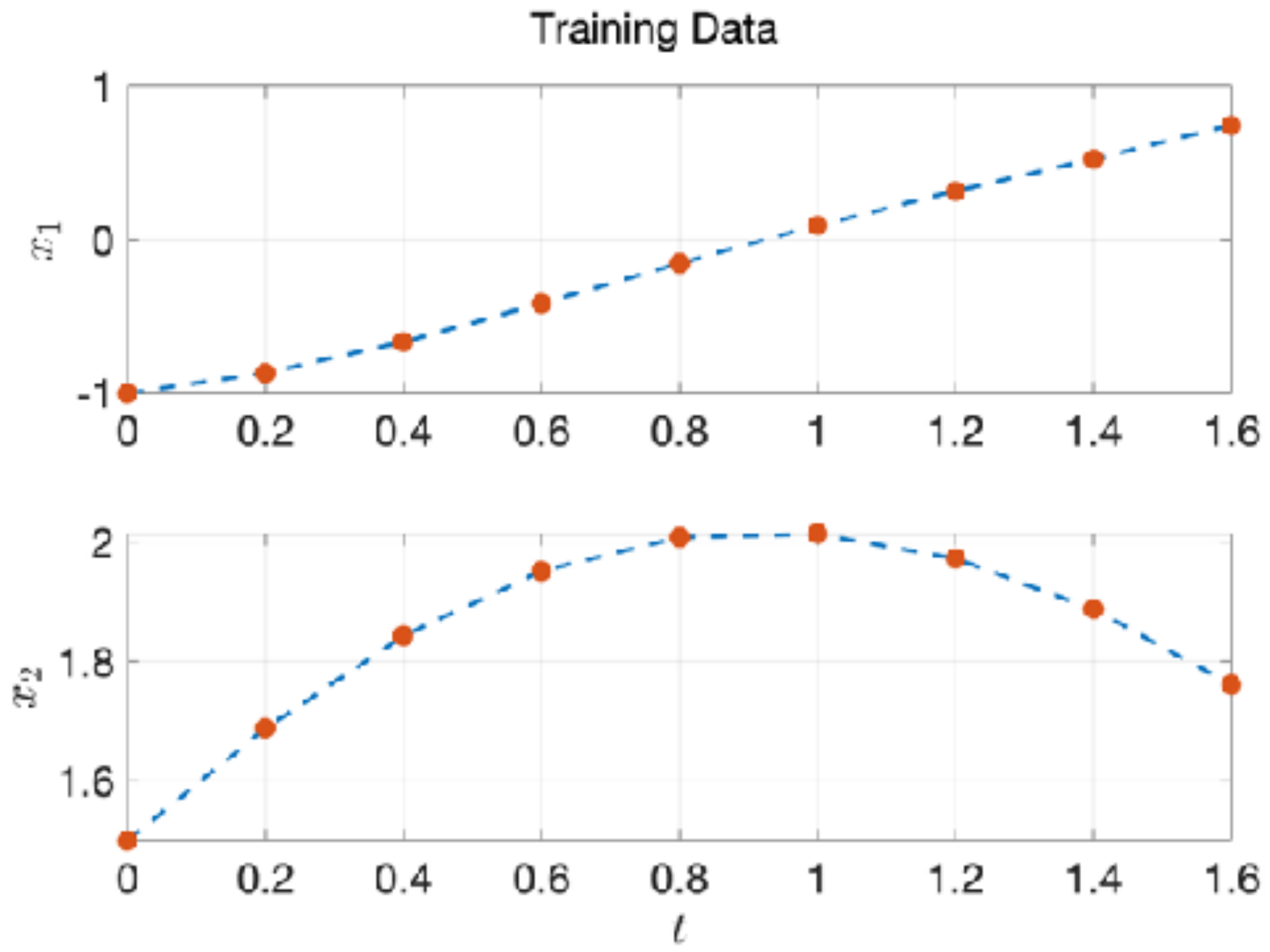
Numerical Example



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1^3 + u \\ -x_1 \end{bmatrix} \qquad Z(x) = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_2^2 & x_1^3 \end{bmatrix}^\top$$

Collecting 9 data points with forcing
$u = -\, sin(t)$, the previous method yields:

$$u = KZ(x) = -\,8.5315x_1 - x_2 - x_1^3$$

$P =$

```
1.0e+08 *

 0.000000002464676   0.000000005247414  -0.000000000656395  -0.000000000386742   0.000000000213237
 0.000000005247414   0.462610683393534   0.334319457507848   0.657146896297293  -0.323265298534464
-0.000000000656395   0.334319457507848   0.497647391617901   0.307730545046117  -0.202924820664369
-0.000000000386742   0.657146896297293   0.307730545046117   1.086556755963401  -0.443877017893051
 0.000000000213237  -0.323265298534464  -0.202924820664369  -0.443877017893051   0.264821810384319
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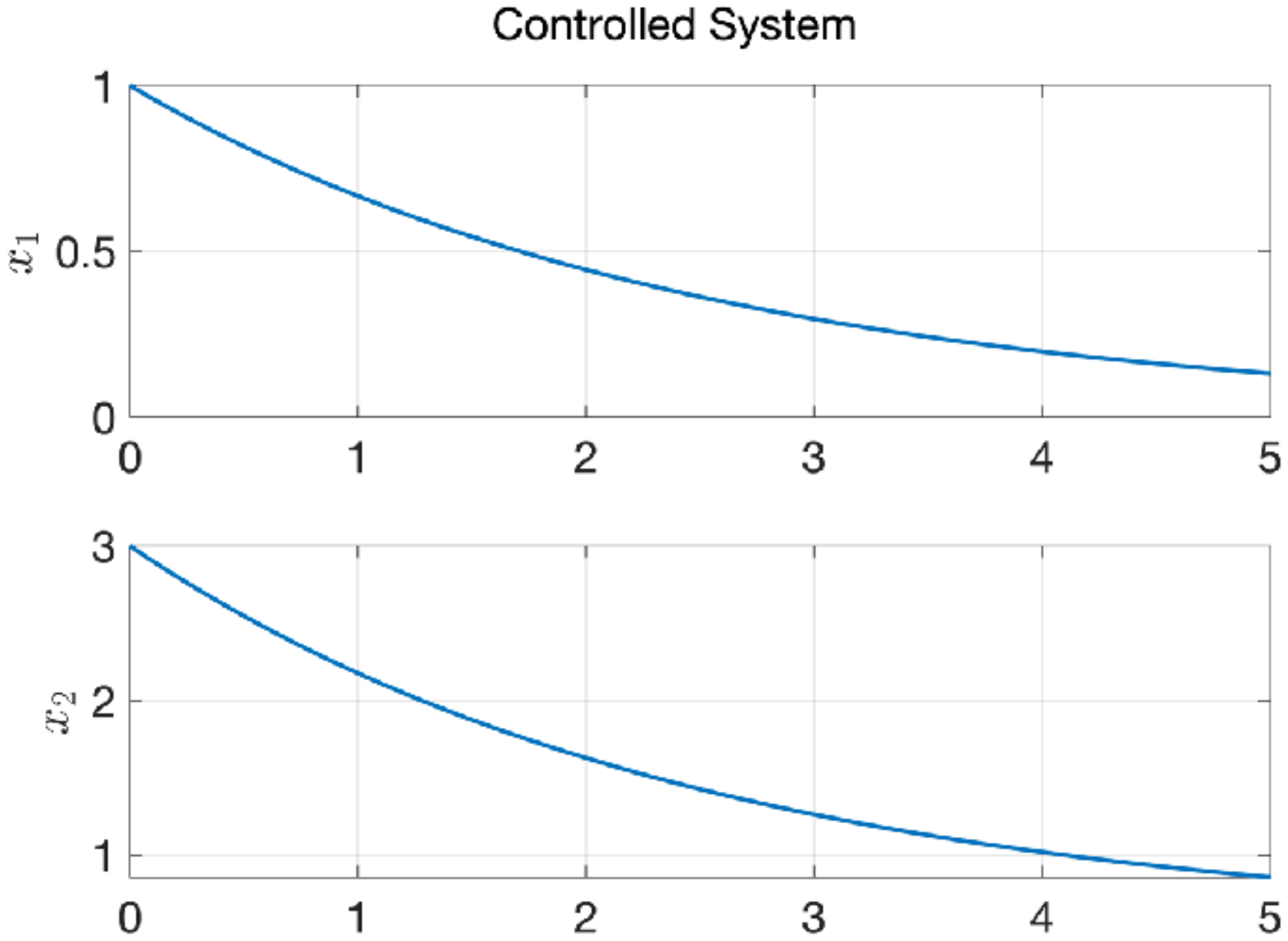

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